Multivortex Solutions of the Abelian Chern-Simons-Higgs Theory

Jooyoo Hong, Yoonbai Kim, and Pong Youl Pac
Department of Physics, Seoul National University, Seoul 151-742, Korea
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We have examined vortex solutions in (2+1)D Chern-Simons-Higgs theory which has no usual Maxwell term. It is shown that the Bogomol'nyi-type equations can be derived for a simple sixth-order Higgs potential and the corresponding general n-vortex solutions should contain $2n$ free parameters. Various characteristics of Chern-Simons vortices are discussed briefly.

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During the last two decades the study of vortices has become an interdisciplinary subject between condensed-matter physics and particle physics. Ginzburg-Landau theory, the macroscopic theory of superconductivity, was known to admit localized solutions of the vortex type.[1] So does the Abelian Higgs model,[2] which is the relativistic extension of the Ginzburg-Landau theory. Characteristically, these vortex solutions carry magnetic flux but are electrically neutral.

Understanding the physics of the recently discovered high-T_c superconductors is one of the most important problems at the moment. As emphasized by several authors, this new superconductor is characterized by its two-dimensional nature, and a P- and T-violating statistical interaction might be important in describing this system. In field-theory language, this P- and T-violating interaction can be related to the Chern-Simons term in (2+1)D Abelian gauge theory, which had been studied by various theorists in other contexts. The basic object in this model behaves like anyons,[3] flux-tube–charged-particle composites with unusual statistics. One usually takes the vortex solution in the Ginzburg-Landau theory as this flux tube. But there is another possibility. With the introduction of the Chern-Simons term in the Abelian Higgs model, it was observed that there also exist vortex solutions. These Chern-Simons vortices are different from the Nielsen-Olesen vortices in that they carry electric charge as well as magnetic flux. Therefore, it is worthwhile to consider the Chern-Simons vortices as another candidate for anyonlike objects.

The solutions studied by the authors of Refs. 4 and 5 are, however, very complicated and therefore it is difficult to check many interesting properties with them. This complication arises from the existence of the Maxwell term, aside from the Chern-Simons term, in the action. Here we would like to point out that to have Chern-Simons vortices, the Maxwell term is not a necessity. It is not unreasonable to consider the theory without the Maxwell term because the Chern-Simons term is dominant over the Maxwell term in the long-distance region (or, equivalently, in the limit of large μ, the coefficient of the Chern-Simons term). Moreover, very recently, Deser and Yang[6] observed that the Higgs mechanism can transmute a nondynamical gauge field into a massive gauge boson. Therefore the dynamics is not lost even in the absence of the Maxwell term provided that there is a Higgs mechanism. As we will show in this paper, the Chern-Simons-Higgs theory does indeed admit mathematically and physically interesting vortex solutions.

There are some special features associated with our Chern-Simons vortex solutions. Above all, the nature of the Bogomol'nyi limit is slightly different from that previously known. While the original Nielsen-Olesen vortex[2] required the scalar potential to be as shown in Fig. 1 with $\lambda = e^2/2$ (λ is the coefficient of the ϕ^4 term), our Chern-Simons vortex demands it to be as shown in Fig. 2, with $h = e^4/8\mu^2$ in the Bogomol'nyi limit[7] (h is the coefficient of the ϕ^6 term) [see Eq. (13) below]. The introduction of the ϕ^6 term in the scalar potential is not unnatural in the light of the renormalizability criterion in 2+1 dimensions. Since ϕ is a complex order parameter, this difference in the shape of the potential (or free energy) may predict a different phenomenology from that of the original Ginzburg-Landau theory. We will briefly discuss some of these aspects at the end.

Let us display some basic characteristics of Chern-

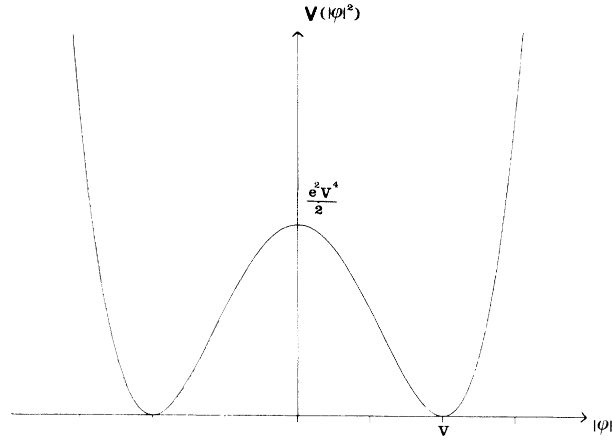

FIG. 1. The shape of the scalar potential for the Nielsen-Olesen–type vortices in the Bogomol'nyi limit.

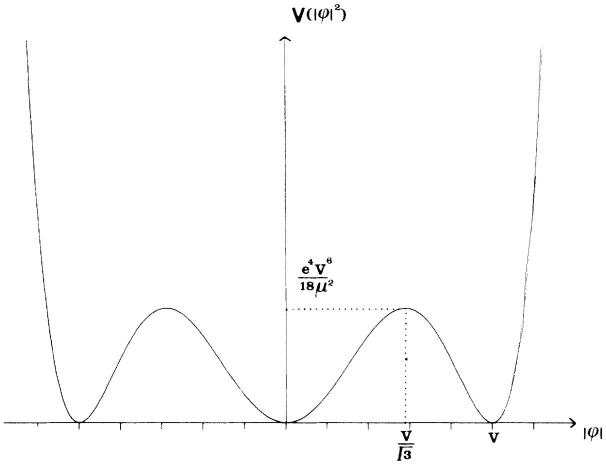

FIG. 2. The shape of the scalar potential for the Chern-Simons vortices in the Bogomol'nyi limit.

Simons vortex solutions first. As stated already, we will dispense with the Maxwell term for electromagnetic fields and thus consider the (2+1)D Chern-Simons-Higgs theory defined by the Lagrangian density[6]

$$\mathcal{L}=\tfrac{1}{4}\mu\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}+\tfrac{1}{2}\,|\partial_\mu-ieA_\mu\phi|^2-V(|\phi|^2)\,. \quad (1)$$

The Euler-Lagrange equations read

$$\tfrac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}=-i\frac{e}{2\mu}(\phi^*\partial_\mu\phi-\partial_\mu\phi^*\phi)-\frac{e^*}{\mu}A_\mu|\phi|^2\,, \quad (2)$$

$$\tfrac{1}{2}(\partial_\mu-ieA_\mu)(\partial^\mu-ieA^\mu)\phi=-\partial V/\partial\phi^*\,. \quad (3)$$

Taking the *Ansatz*[4,5] for the n-vortex solution as

$$\mathbf{A}(\rho)=-\hat{\boldsymbol{\theta}}A(\rho)/\rho\,,$$
$$\phi(\rho)=f(\rho)e^{-in\theta}\,, \quad (4)$$

the above equations of motion reduce to

$$\frac{1}{\rho}\frac{dA}{d\rho}+\frac{e^2}{\mu}f^2A_0=0\,, \quad (5a)$$

$$\rho\frac{dA_0}{d\rho}+\frac{e^2}{\mu}f^2\left(\frac{n}{e}+A\right)=0\,, \quad (5b)$$

$$\frac{d^2f}{d\rho^2}+\frac{1}{\rho}\frac{df}{d\rho}-\frac{e^2}{\rho^2}f\left(\frac{n}{e}+A\right)^2+e^2fA_0^2=\frac{\partial V}{\partial f}\,. \quad (5c)$$

The appropriate boundary conditions are

$$\lim_{\rho\to\infty}A(\rho)=-n/e,\quad \lim_{\rho\to\infty}f(\rho)=v\,,$$
$$\lim_{\rho\to 0}A(\rho)=0,\quad \lim_{\rho\to 0}f(\rho)=0\,. \quad (6)$$

We do not have exact analytic solutions for these coupled differential equations. However, it is not difficult to find asymptotic solutions: In fact, for large ρ,

$$A_0(\rho)\simeq CK_0\left(\frac{e^2v^2}{|\mu|}\rho\right),$$
$$A(\rho)\simeq-\frac{n}{e}-\frac{\mu}{e^2v^2}C\rho\frac{d}{d\rho}K_0\left(\frac{e^2v^2}{|\mu|}\rho\right), \quad (7)$$

where K_0 is the modified Bessel function and C is a constant. In the other limit, $\rho\to 0$, the various functions should behave as

$$f(\rho)\to f_0\rho^{|n|}+\cdots\,,$$
$$A_0(\rho)\to \alpha_0-\frac{ef_0^2n}{2\mu|n|}\rho^{2|n|}+\cdots\,, \quad (8)$$
$$A(\rho)\to -\frac{e^2f_0^2\alpha_0}{2\mu(|n|+1)}\rho^{2|n|+2}+\cdots\,,$$

where α_0 and f_0 are constants.

The magnetic flux and the electric charge carried by this vortex are easily determined on the basis of Eqs. (5) and (6), i.e.,

$$\Phi=-\int d^2x\,F_{12}=n(2\pi/e)\,,$$
$$Q=\int d^2x\,j_0=n(2\pi\mu/e)\,. \quad (9)$$

Note the relationship $Q=\mu\Phi$. These kinds of charged-vortex solutions have also been found by the authors of Refs. 4 and 5 in the Abelian Higgs model with both the Maxwell term and the Chern-Simons term. Their solutions are similar in nature with ours. But we should stress again that there is no need to have the Maxwell term in order for the theory to admit vortex solutions. The only significant change caused by the Maxwell term is the behavior of the vortex solution in the $\rho\to 0$ limit; observe that, in our case, the magnetic field approaches zero (and not some constant value) as $\rho\to 0$.

The Chern-Simons vortex solutions discussed above can be interpreted as n vortices superimposed at the same point. Then it would be very interesting to also look for the possibility of having static configurations of n vortices arbitrarily distributed on a plane. For Nielsen-Olesen–type neutral vortices, it is known that this happens in the so-called Bogomol'nyi limit, as discussed by Weinberg,[8] Jacobs and Rebbi,[9] and Taubes.[10] In this limit which involves a special choice of Higgs coupling ($\lambda=e^2/2$), the vortices behave like noninteracting particles. We will now show that an analogous situation can also be found for the Chern-Simons-Higgs theory.

The energy-momentum tensor $T_{\mu\nu}$ for our theory is given as

$$T_{\mu\nu}=\tfrac{1}{2}(\partial_\mu+ieA_\mu)\phi^*(\partial_\nu-ieA_\nu)\phi+\tfrac{1}{2}(\partial_\nu+ieA_\nu)\phi^*(\partial_\mu-ieA_\mu)\phi-\eta_{\mu\nu}[\tfrac{1}{2}(\partial_\alpha+ieA_\alpha)\phi^*(\partial^\alpha-ieA^\alpha)\phi-V(|\phi|^2)] \quad (10)$$

with no contribution from the Chern-Simons term. Thus the energy functional for static field configurations reads

$$E=\int d^2x[\tfrac{1}{2}\,|(\partial_i-ieA_i)\phi|^2+\tfrac{1}{2}e^2A_0^2|\phi|^2+V(|\phi|^2)]\,. \tag{11}$$

Following Ref. 7, one may rearrange the terms and perform the partial integration to get

$$\begin{aligned}E=|n|\pi v^2+\int d^2x\Bigg[&\tfrac{1}{2}[(\partial_1\phi_1+eA_1\phi_2)\mp(\partial_2\phi_2-eA_2\phi_1)]^2+\tfrac{1}{2}[(\partial_2\phi_1+eA_2\phi_2)\pm(\partial_1\phi_2-eA_1\phi_2)]^2\\&+\frac{\mu^2}{2e^2(\phi_1^2+\phi_2^2)}\left(F_{12}\pm\frac{e^3}{2\mu^2}(\phi_1^2+\phi_2^2)(v^2-\phi_1^2-\phi_2^2)\right)^2\\&+\left(V(|\phi|^2)-\frac{e^4}{8\mu^2}(\phi_1^2+\phi_2^2)(v^2-\phi_1^2-\phi_2^2)^2\right)\Bigg]\,,\end{aligned} \tag{12}$$

where the upper (lower) sign refers to positive (negative) vortex number and the field A_0 has been eliminated by using the constraint equation following from the equations of motion.

Clearly, the energy minimum $E_{\min}=|n|\pi v^2$ can be achieved if and only if the potential is of the form

$$V(|\phi|^2)=h|\phi|^2(|\phi|^2-v^2)^2 \tag{13}$$

with the critical value of $h=e^4/8\mu^2$. This precisely gives the Bogomol'nyi limit. In this limit, the lower bound is saturated if and only if the following Bogomol'nyi equations are satisfied:

$$\begin{aligned}&(\partial_1\phi_1+eA_1\phi_2)\mp(\partial_2\phi_2-eA_2\phi_1)=0\,,\\&(\partial_2\phi_1+eA_2\phi_2)\pm(\partial_1\phi_2-eA_1\phi_1)=0\,,\\&F_{12}\pm(e^3/2\mu^2)(\phi_1^2+\phi_2^2)(v^2-\phi_1^2-\phi_2^2)=0\,.\end{aligned} \tag{14}$$

Notice that, in this Bogomol'nyi limit, the other diagonal components of the energy-momentum tensor also vanish; i.e., $T_{11}=T_{22}=0$. It is still quite difficult to obtain analytic solutions of Eq. (14). Nevertheless, upon expressing the scalar field as

$$\phi=ve^{f_1+if_2}\quad(f_1,f_2\text{ are real functions})\,, \tag{15}$$

the problem of solving Eq. (14) reduces to the analysis of the nonlinear partial differential equation

$$\nabla^2 f_1+(e^4v^4/2\mu^2)e^{2f_1}(e^{2f_1}-1)=0\,, \tag{16}$$

which closely resembles the form of the equation known for the Nielsen-Olesen vortex case. Analyses similar to those used by Taubes[10] might provide some useful information regarding the solutions of Eq. (16).

The general n-vortex solution of Eq. (14) should involve $2n$-independent free parameters, the same number as needed to specify the positions of n vortices. This fact can be established by counting the zero modes of the small-fluctuation equations about any specific n-vortex solution of Eq. (14). Actually, together with the Coulomb gauge condition, the linear fluctuation equations arising from Eq. (14) form a pair of 2D Dirac equations and the number of independent zero modes can then be determined by the usual index theorem. The entire proof can be taken verbatim from the paper of Weinberg[8] and hence shall not be repeated here.

Finally, we note that the angular momentum carried by our Chern-Simons vortices can easily be calculated with the help of the equations of motion given in Eq. (5). After some straightforward algebra, we find

$$J=\int d^2x\,\epsilon^{ij}x_iT_{0j}=\frac{Q\Phi}{4\pi}\left[=\pi\mu\left(\frac{n}{e}\right)^2\right]. \tag{17}$$

Now suppose that the Chern-Simons term is induced dynamically as the vacuum polarization effect of a single Dirac fermion of charge q and mass m_f. Then the value of the topological mass is given by[11]

$$\mu=\frac{m_f}{|m_f|}\frac{q^2}{4\pi}\,. \tag{18}$$

In this case the charge and the angular momentum of a single vortex of vorticity n will be

$$Q=(q^2/2e)n,\quad J=(nq/2e)^2 \tag{19}$$

up to the sign proportional to $m_f/|m_f|$. If the scalar field carries twice the charge of the fermion (as in the case of a Cooper pair), the electric charge and the angular momentum become fractional,

$$Q=\tfrac{1}{4}qn,\quad J=\tfrac{1}{16}n^2\,. \tag{20}$$

This kind of argument may have some useful applications in condensed-matter physics.

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Note added.—After finishing our analysis, we were informed by Professor Choonkyu Lee that essentially the same conclusion concerning Chern-Simons vortices has been reached by Jackiw and Weinberg.[12]

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